Stewartson's compressibility correlation in three dimensions

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On the assumption of (*a*) zero heat transfer, (*b*) unit Prandtl number, (*c*) linear viscosity temperature relation, and (*d*) small cross-flow, a correlation between a compressible laminar boundary-layer flow and an incompressible flow with different external conditions is established.

The correlation is applied to a known approximate method of solution for incompressible flow. For flow over a slender wing it is found that, so far as the effect on limiting streamlines—and hence on separation—is concerned, the fluid behaves as though it were incompressible but with pressure gradients multiplied by $1+\frac{1}{2}(\gamma-1)M_0^2$ approximately, where M_0 is the Mach number, supposed to be moderate in value. The effect on drag is also estimated.

An attempt is made to assess the effect of variable Prandtl number Pr, and it is found that the above multiplying factor must be replaced by

$$1+\tfrac{1}{2}Pr^{\frac{1}{2}}(\gamma-1)M_0^2,$$

but that variations in Prandtl number have little effect on the previous estimate of drag.

1. Introduction

Stewartson (1949) showed that if it is assumed in two-dimensional flow that (*a*) the surface is thermally insulating, (*b*) viscosity varies as the absolute temperature, and (*c*) the Prandtl number is unity, it is possible to transform the co-ordinates so that the boundary-layer equations for a compressible fluid with a given main-stream velocity become identical with those for an incompressible fluid with a different main-stream velocity. The particular transformation used by Stewartson was also used by Illingworth (1949), and is closely allied to one due to Howarth (1948).

In general three-dimensional flow it does not seem possible to find a transformation to correlate compressible and incompressible boundary layers in this way. However, there is one case which is amenable, namely the case of small cross-flow which has been fairly extensively studied for incompressible fluids. Here it is assumed that the velocity in the boundary layer normal to the external streamlines and certain of its derivatives are small. A general discussion of this case was given by Cooke (1959*a*) who derived the equations of motion in a 'streamline' co-ordinate system in which the co-ordinate curves were the projections on to the surface of the external streamlines and their orthogonal trajectories in the surface.

We show in this paper that the same transformation, simply extended, will correlate the compressible boundary layer associated with a given external flow with an incompressible boundary layer associated with a different external flow, in the case where Stewartson's conditions (*a*), (*b*) and (*c*) hold, together with the small cross-flow condition. In the present state of knowledge the latter restriction does not trouble us unduly, since it is found in most approximate methods for incompressible flow.

The correlation having been established, a study is made of the flow over a conically cambered delta wing for which an incompressible-flow calculation has already been made. The effect is best seen by examining the angle β between streamlines and limiting streamlines as shown in figure 1. This angle is considerably increased by compressibility, and this leads to an earlier separation. It will be found that a useful guide to the effect on β is to assume the flow to be the same as it would be with incompressible fluid, but with all pressure gradients multiplied by $1+\frac{1}{2}(\gamma-1)M_0^2$, where M_0 is the Mach number. In the case studied, M_0 is equal to 2 and γ equal to 1·4, so that this multiplier is equal to 1·8.

An approximate allowance can be made for a Prandtl number Pr not equal to unity. A modified transformation, due to Rott (1953), is used. This causes minor modifications all through; in particular it is found that the factor mentioned above is changed to $1+\frac{1}{2}Pr^{\frac{1}{2}}(\gamma-1)M_0^2$ approximately, so that the multiplier 1·8 is reduced to 1·68 when M_0 is equal to 2. The other effects of the change in Prandtl number are very small for moderate Mach numbers.

2. The equations of motion

Taking streamline co-ordinates, we may define the length element dl by the relation

$$dl^2 = h_1^2 d\xi^2 + h_2^2 d\eta^2 + d\zeta^2, \tag{1}$$

where ζ is distance normal to the surface, while the curves $\eta = \text{const.}$, $\zeta = 0$ are the projections of the external streamlines on to the surface, and $\xi = \text{const.}$, $\zeta = 0$ are their orthogonal trajectories. As usual, h_1 and h_2 are taken to be independent of ζ. u, v and w are velocity components in the co-ordinate directions.

Cooke (1959*a*) wrote $h_2 = r$ and let ds be the length element along the curve $\eta = \text{const.}$, $\zeta = 0$; that is $ds = h_1 d\xi$. Assuming small cross-flow, Cooke obtained equations of motion which, when the thermal conductivity k was replaced by $c_p\mu$ (for unit Prandtl number), became

$$\rho\left(u\frac{\partial u}{\partial s} + w\frac{\partial u}{\partial \zeta}\right) = \rho_e u_e \frac{\partial u_e}{\partial s} + \frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right), \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial s} + w\frac{\partial v}{\partial \zeta} + \frac{uv}{r}\frac{\partial r}{\partial s}\right) = \kappa(\rho_e u_e^2 - \rho u^2) + \frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right), \tag{3}$$

$$\rho c_p\left(u\frac{\partial T}{\partial s} + w\frac{\partial T}{\partial \zeta}\right) = -\rho_e u u_e \frac{\partial u_e}{\partial s} + \frac{\partial}{\partial \zeta}\left(c_p\mu\frac{\partial T}{\partial \zeta}\right) + \mu\left(\frac{\partial u}{\partial \zeta}\right)^2, \tag{4}$$

$$\frac{\partial}{\partial s}(\rho r u) + \frac{\partial}{\partial \zeta}(\rho r w) = 0, \tag{5}$$

where $$\kappa = -\frac{1}{h_1 r}\frac{\partial h_1}{\partial \eta}.$$

We take the suffix e to denote values just outside the boundary layer, and the suffix 0 to denote values at some isentropic reference position.

Stewartson (1949) used the relation

$$\mu/\mu_0 = T/T_0;$$

but, following Chapman & Rubesin (1949), we may replace this by

$$\mu/\mu_0 = C(T/T_0). \tag{6}$$

In the special case of zero heat transfer and unit Prandtl number, a solution (Moore, 1951) of the energy equation is Crocco's temperature relation

$$\frac{T}{T_e} = 1 + \frac{\gamma-1}{2a_e^2}(u_e^2 - u^2 - v^2), \tag{7}$$

where a_e is the velocity of sound just outside the boundary layer. As v is small, we may neglect v^2 in equation (7).

3. The transformation

Modifying Stewartson's transformation slightly, we write

$$Z = \frac{a_e}{a_0}\int_0^\zeta \frac{\rho}{\rho_0}\, d\zeta, \tag{8}$$

$$S = C\int^s \left(\frac{a_e}{a_0}\right)^{(3\gamma-1)/(\gamma-1)} ds. \tag{9}$$

We write also $$U_e = a_0 u_e/a_e, \tag{10}$$

$$V = v, \tag{11}$$

$$K = \frac{\kappa}{C}\left(\frac{a_0}{a_e}\right)^{(4\gamma-1)/(\gamma-1)} q, \tag{12}$$

$$q = 1 + \tfrac{1}{2}(\gamma-1)M_0^2. \tag{13}$$

We omit the details of the analysis, which are given in the Appendix. It is shown there that equations (2) and (3) become

$$U\frac{\partial U}{\partial S} + W\frac{\partial U}{\partial Z} = U_e\frac{\partial U_e}{\partial S} + \nu_0\frac{\partial^2 U}{\partial Z^2}, \tag{14}$$

$$U\frac{\partial V}{\partial S} + W\frac{\partial V}{\partial Z} + \frac{UV}{r}\frac{\partial r}{\partial S} = K(U_e^2 - U^2) + \nu_0\frac{\partial^2 V}{\partial Z^2}, \tag{15}$$

where W satisfies $$\frac{\partial}{\partial S}(rU) + \frac{\partial}{\partial Z}(rW) = 0. \tag{16}$$

These equations are the same in form as equations (2), (3) and (5), with ρ and μ constant and $\nu_0 = \mu/\rho$. The boundary conditions $u = v = w = 0$ when $\zeta = 0$, and $u = u_e$, $v = 0$ when $\zeta = \infty$ become $U = V = W = 0$ when $Z = 0$ and $U = U_e$, $V = 0$ when $Z = \infty$.

Thus we have reduced the compressible equations with small cross-flow to incompressible equations with small cross-flow, and there is a complete correlation. Note that, although the co-ordinates are changed, r is not changed.

4. Arbitrary surface temperature

Stewartson (1949) extended the use of his transformation to the case where the boundary condition on the surface is arbitrary. This does not yield a correlation, but the transformation greatly simplifies the equations. We include this case for completeness.

Stewartson wrote

$$\frac{T}{T_e} = \left\{1 + \frac{\gamma - 1}{2a_e^2}(u_e^2 - u^2)\right\} + \frac{T_0}{T_e}\{1 + \tfrac{1}{2}(\gamma - 1) M_0^2\} A, \tag{17}$$

where A is the temperature function. Note that we must have $A = 0$ when $\zeta = \infty$; when $\zeta = 0$, A is determined by the specified surface temperature T_w. Exactly as in Stewartson's paper, this leads to

$$U\frac{\partial U}{\partial S} + W\frac{\partial U}{\partial Z} = U_e \frac{\partial U_e}{\partial S}(1 + A) + \nu_0 \frac{\partial^2 U}{\partial Z^2},$$

in place of equation (14).

Now, equation (17) may be written

$$\frac{T}{T_e} = B + \frac{T_0}{T_e} q A,$$

where B satisfies the energy equation (4), as is clear from equation (7) with v^2 neglected. It is now quite straightforward to show that A satisfies the equation

$$U\frac{\partial A}{\partial S} + W\frac{\partial A}{\partial Z} = \nu_0 \frac{\partial^2 A}{\partial Z^2}, \tag{18}$$

with boundary conditions $A = 0$ when $Z = \infty$ and A given by the specified surface temperature when $Z = 0$.

Finally, equation (15) is changed to

$$U\frac{\partial V}{\partial S} + W\frac{\partial V}{\partial Z} + \frac{UV}{r}\frac{\partial r}{\partial S} = K\{U_e^2(1 + A) - U^2\} + \nu_0 \frac{\partial^2 V}{\partial Z^2}.$$

5. Application of incompressible approximate methods

We now apply the transformation procedure to the methods of Zaat (1956, 1957) and Cooke (1959*b*). In these the length element is defined by

$$dl^2 = \frac{d\phi^2}{u_e^2} + \frac{d\psi^2}{\bar{\rho} u_e^2},$$

and this we will now suppose to be transformed into

$$dL^2 = \frac{d\Phi^2}{U_e^2} + \frac{d\Psi^2}{\bar{P} U_e^2}.$$

Since r is unchanged in the transformation, we have

$$\frac{1}{\bar{\rho}u_e^2} = r^2 = \frac{1}{\bar{P}U_e^2},$$

and so
$$\bar{P} = \bar{\rho}(a_e^2/a_0^2). \tag{19}$$

We also note that
$$\kappa = \rho^{\frac{1}{2}}\,\partial u_e/\partial\psi.$$

The streamwise length element is transformed by equation (9). Hence, when ψ or Ψ are constant, we have

$$\frac{d\Phi}{U_e} = C\left(\frac{a_e}{a_0}\right)^{(3\gamma-1)/(\gamma-1)}\frac{d\phi}{u_e}$$

or, by equation (10),
$$d\Phi = C\left(\frac{a_e}{a_0}\right)^{2\gamma/(\gamma-1)} d\phi. \tag{20}$$

The transformation of κ in equation (12) is used to find how ψ is transformed. It leads to

$$d\Psi = C\left(\frac{a_e}{a_0}\right)^{2\gamma/(\gamma-1)} d\psi, \tag{21}$$

when ϕ or Φ are constant. To verify this, we have, noting that $U_e(\partial U_e/\partial\Psi)$ is given by equation (A 5) with $\bar{S}$ replaced by Ψ,

$$\begin{aligned} K = \bar{P}^{\frac{1}{2}}\frac{\partial U_e}{\partial \Psi} &= \frac{\bar{\rho}^{\frac{1}{2}}}{C}\left(\frac{a_0}{a_e}\right)^{(4\gamma-2)/(\gamma-1)}\frac{\partial u_e}{\partial\psi}q \\ &= \frac{\kappa}{C}\left(\frac{a_0}{a_e}\right)^{(4\gamma-2)/(\gamma-1)} q. \end{aligned}$$

This, by equation (12), is the right form for K, and thus the relation (21) is verified.

We may therefore follow Zaat's or Cooke's methods using the new variables according to the above rules. We shall only go into details in connexion with the latter method. It is, however, just as easy to apply the transformation to Zaat's method.

In the transformed (incompressible) flow, we shall denote all variables by capital letters, and shall call them 'transformed variables', as distinct from 'true variables' in the compressible flow, which are denoted in general by the corresponding small letters.

Cooke's equations in transformed variables are

$$\frac{\partial}{\partial\Phi}\left(\frac{U_e^4\Sigma}{\bar{P}}\right) = 5{\cdot}08\frac{U_e^2}{\bar{P}}, \tag{22}$$

$$\bar{P}\Sigma^{\frac{1}{2}}\frac{\partial}{\partial\Phi}\left(\frac{\Sigma^{\frac{1}{2}}\Theta_{21}}{\bar{P}}\right) = \frac{1}{U_e^2}\{\Pi^* + M^*(0{\cdot}067\Lambda^* - 0{\cdot}669)\}, \tag{23}$$

where
$$\Theta_{21} = -(0{\cdot}295 + 0{\cdot}022\Lambda^*)\,\Pi^* - (0{\cdot}030 + 0{\cdot}004\Lambda^*)\,M^*, \tag{24}$$

$$\Lambda^* = \Sigma U_e(\partial U_e/\partial\Phi), \tag{25}$$

$$M^* = \bar{P}^{\frac{1}{2}}\Sigma U_e(\partial U_e/\partial\Psi). \tag{26}$$

In these equations, Π^* is a boundary-layer velocity-profile parameter, Σ is a variable proportional to the square of the boundary-layer thickness, Λ^* and M^* are proportional to the streamwise and crosswise pressure gradients, and Θ_{21} is a mixed 'momentum thickness'. [In Cooke (1959*b*) what we would here denote by λ^*, μ^*, π^* were written Λ, M and Π, and Θ_{21} had a different meaning. Nevertheless, the capital-letter notation is so convenient that we shall continue to use it. This should cause no confusion in practice.]

Introducing true variables (except for Σ) and putting $\gamma = 1{\cdot}4$, we find by equations (10), (19) and (20) that equation (22) becomes

$$\frac{\partial}{\partial \phi}\left\{\left(\frac{a_0}{a_e}\right)^6 \frac{u_e^4}{\bar{\rho}}\left(\frac{\Sigma}{C}\right)\right\} = 5{\cdot}08\left(\frac{a_e}{a_0}\right)^3 \frac{u_e^2}{\bar{\rho}}, \tag{27}$$

which thus determines Σ/C as a function of the true co-ordinates.

Equations (25) and (26) become, using equations (10), (19), (20) and (21) and putting $\gamma = 1{\cdot}4$,

$$\Lambda^* = \left(\frac{\Sigma}{C}\right)\left(\frac{a_0}{a_e}\right)^{11} u_e \frac{\partial u_e}{\partial \phi} q, \tag{28}$$

$$M^* = \bar{\rho}^{\frac{1}{2}}\left(\frac{\Sigma}{C}\right)\left(\frac{a_0}{a_e}\right)^{10} u_e \frac{\partial u_e}{\partial \psi} q. \tag{29}$$

Thus, Λ^* and M^* are determined as functions of the true co-ordinates, since Σ/C is known by equation (27).

Equation (23) may be written

$$\bar{\rho}\left(\frac{\Sigma}{C}\right)^{\frac{1}{2}}\left(\frac{a_0}{a_e}\right)^7 \frac{\partial}{\partial \phi}\left\{\frac{(\Sigma/C)^{\frac{1}{2}}\,\Theta_{21}}{\bar{\rho}}\frac{a_0^2}{a_e^2}\right\} = \frac{1}{u_e^2}\{\Pi^* + M^*(0{\cdot}067\Lambda^* - 0{\cdot}669)\}; \tag{30}$$

and hence, with the aid of equation (24), Π^* and Θ_{21} can be found as functions of the true co-ordinates. The method of solution is exactly the same as that outlined by Cooke (1959*b*).

For the skin friction we have

$$\tau_{01} = \mu_w \left(\frac{\partial u}{\partial \zeta}\right)_0;$$

hence, using the relations

$$T_w/T_e = \rho_e/\rho_w, \quad u/u_e = U/U_e,$$

and the isentropic relations

$$\rho_e/\rho_0 = (a_e/a_0)^5, \quad T_e/T_0 = (a_e/a_0)^2,$$

we obtain, following Cooke (1959*b*),

$$\tau_{01} = C\mu_0 \frac{u_e}{(\Sigma\nu_0)^{\frac{1}{2}}}\left(\frac{a_e}{a_0}\right)^8 \frac{2}{3\pi^{\frac{1}{2}}}(2+\Lambda^*). \tag{31}$$

We find in a similar way

$$\tau_{02} = C\mu_0 \frac{u_e}{(\Sigma\nu_0)^{\frac{1}{2}}}\left(\frac{a_e}{a_0}\right)^7 \frac{2}{3\pi^{\frac{1}{2}}}\left(\frac{3\pi^{\frac{1}{2}}}{2}\Pi^* + M^*\right). \tag{32}$$

If β is the angle between streamlines and limiting streamlines, we have

$$\tan\beta = \frac{\tau_{02}}{\tau_{01}} = \frac{(3\pi^{\frac{1}{2}}/2)\,\Pi^* + M^*}{2+\Lambda^*}\left(\frac{a_0}{a_e}\right). \tag{33}$$

6. Effect of arbitrary Prandtl number

Rott (1953) has suggested that, if the Prandtl number Pr is not unity, equation (7) may be replaced by

$$\frac{T}{T_e} = 1+\bar{r}\left(\frac{\gamma-1}{2a_e^2}\right)(u_e^2-u^2-v^2), \tag{34}$$

where $\bar{r}$ is the recovery factor, which is a function of Prandtl number only. For values of σ not too far from unity, the approximate value $\bar{r} = Pr^{\frac{1}{2}}$ is usually taken.

Equation (34) is exact for $Pr = 1$. When $Pr \neq 1$, it still satisfies the boundary conditions at the wall and at the edge of the boundary layer. Consequently, equation (34) is probably a reasonable approximate solution of the energy equation (4).

Rott (1953) pointed out that in two dimensions a correlation can still be found by modifying Stewartson's transformation. In the present notation the new transformation is

$$Z = \left(\frac{a_e}{a_0}\right)^{\bar{r}}\int_0^{\zeta}\frac{\rho}{\rho_0}\,d\zeta,$$

$$S = C\int^{s}\left(\frac{a_e}{a_0}\right)^{2\gamma/(\gamma-1)+\bar{r}}ds,$$

$$U_e = (a_0/a_e)^{\bar{r}}\,u_e.$$

This reduces to the earlier transformation when $\bar{r} = 1$.

It is quite easy to show that the correlation in the V equation will still hold if, in addition to the above changes, equation (12) and (13) are replaced by

$$K = \frac{\kappa}{C}\left(\frac{a_0}{a_e}\right)^{(4\gamma-1)/(\gamma-1)}Q,$$

$$Q = 1+\tfrac{1}{2}\bar{r}(\gamma-1)\,M_0^2+(1-\bar{r})\{(a_e/a_0)^2-1\}. \tag{35}$$

This causes minor modifications to the approximate equations of § 5. These are given in table 1. We omit the details of the calculations.

7. An example and discussion

We consider the flow past the upper surface of a thin conically cambered delta wing with attached flow at the leading edges, as described by Brebner (1957). This wing (see figure 1) is flat inboard of the shoulders, which are straight lines through the apex, their equations in the notation of figure 1 being $y/Kx = \pm 0{\cdot}6$. They are shown dotted in the figure. Outboard of the shoulders the wing is drooped in such a way as to make the downwash distribution parabolic. The design lift coefficient chosen is 0·1; this will be achieved at a certain angle of incidence which will give attached flow along the leading edges. The fact that the calculated separation line is $y/Kx = 0{\cdot}63$ shows that the flow is unable to go right round the shoulder; hence, unless the flow quickly re-attaches after separating, the boundary-layer solution and the external flow cannot match, since the external velocity components used in the solution were calculated on the basis of unseparated flow. Nevertheless, the example will still illustrate the method (even though the external flow used is not that over the wing specified) and will show the general effect of compressibility.

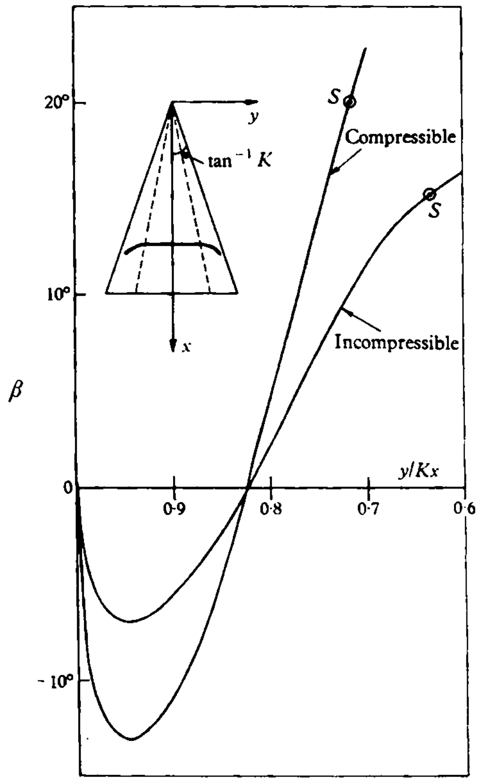

FIGURE 1. Angle between streamlines and limiting streamlines. Mach 2. ⊙ S, points of separation.

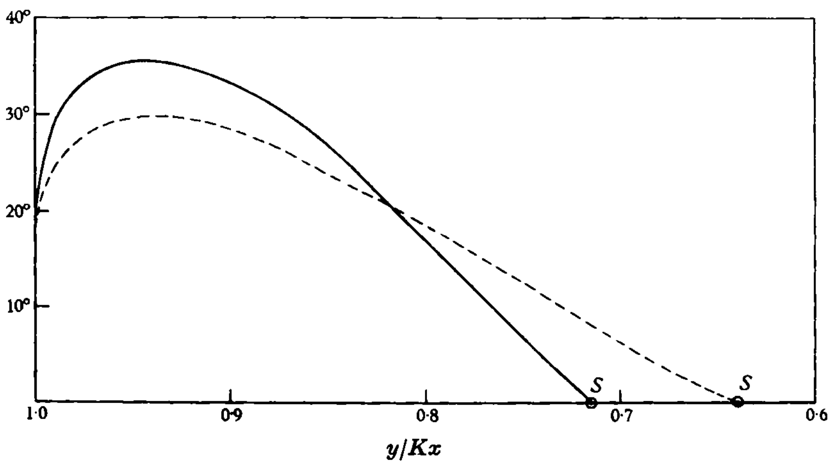

FIGURE 2. Angle between limiting streamlines and rays. ——, Compressible; ----, incompressible; ⊙ S, points of separation.

The boundary layer over this wing was the subject of an incompressible-flow calculation by Cooke (1959*b*). This has now been recalculated, using the same external flow but with $M_0 = 2$, the suffix 0 referring to conditions at infinity. The flow is conical, so that calculations along one streamline are sufficient to determine the flow over the whole wing as far as the separation line.

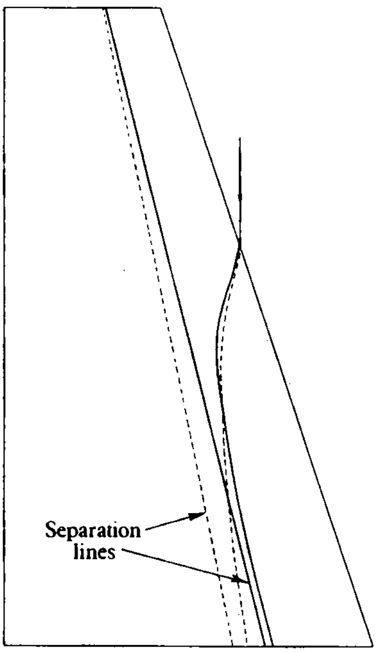

FIGURE 3. Limiting streamlines and separation lines. ——, Compressible; ----, incompressible.

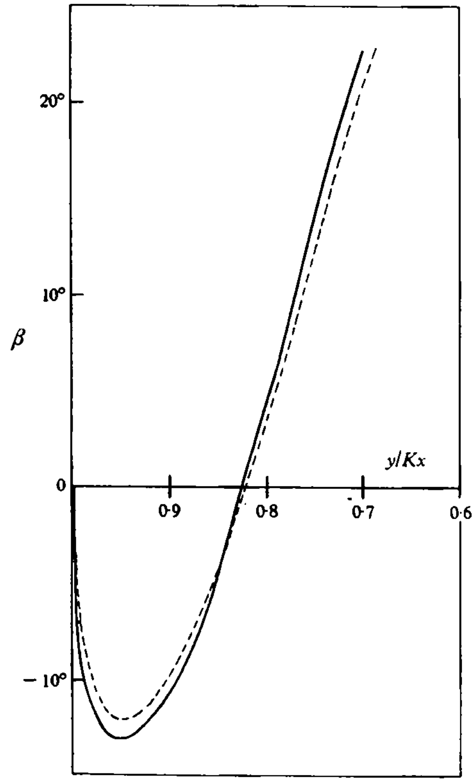

FIGURE 4. Angle between streamlines and limiting streamlines. ——, Full; ----, simplified.

The effect of compressibility is best seen by plotting the angle β between streamlines and limiting streamlines, as in figure 1. The effect is clearly seen to be an increase in this angle at all positions. Figure 2 shows the angle between limiting streamlines and rays. Separation takes place when this angle vanishes. It is earlier than in the incompressible case. In figure 3 an attempt is made to draw part of a limiting streamline and a separation line for the two cases.

It we are dealing with moderate Mach numbers, such as $M_0 = 2$, the ratio a_0/a_e is not far from unity. If it were constantly equal to unity, then Σ/C, obtained from equation (27), would have the same value as σ calculated on the assumption of incompressibility. Λ^* and M^* would then take their incompressible values multiplied by q, as can be seen from equations (28) and (29). Moreover, if we could assume $a_0/a_e = 1$ and Λ^* not too large, we should find from equations (23) and (24) that Π^* would also take its incompressible value multiplied by q.

Figures 4 and 5 shows curves for the compressible case marked 'full' and 'simplified'. The former are obtained by the full use of equations (27), (30), (31), (32) and (33). The latter are obtained from incompressible calculations followed by the multiplication of Λ^*, M^* and Π^* by $q\{= 1+\frac{1}{2}(\gamma-1)M_0^2\}$ and then by the use of equations (31), (32) and (33). It will be seen that the simplified method gives results of fair accuracy. Moreover, the simplified method shows that, as far as limiting streamline direction is concerned, compressible flow is like incompressible flow with pressure gradients multiplied by $1+\frac{1}{2}(\gamma-1)M_0^2$, at any rate for moderate Mach numbers. This effect is large; for example the pressure gradients would be doubled for $M_0 = \sqrt{5}$.

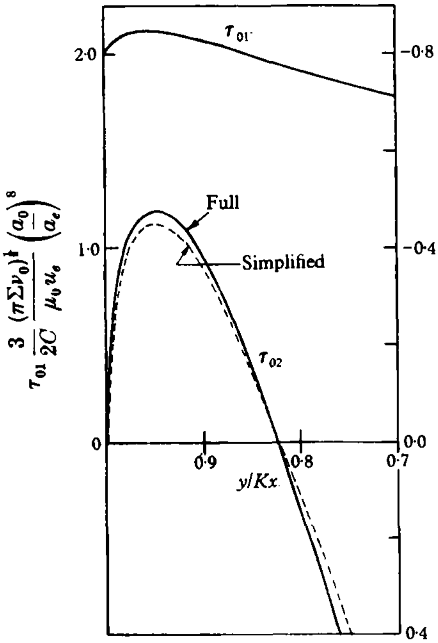

FIGURE 5

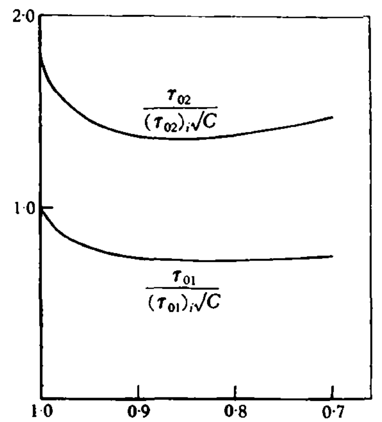

FIGURE 6

FIGURE 5. Values of streamwise and crosswise skin friction, τ_{01} and τ_{02}. Compressible, upper surface. ——, Full; ----, simplified. (In the curve for τ_{01}, the results for the full and simplified methods are indistinguishable.)

FIGURE 6. Ratios of compressible to incompressible skin friction components. Upper surface.

The effect of compressibility on drag comes mainly from the term $(a_e/a_0)^8$ in equation (31), while the factor q multiplying the pressure gradient has only a small effect. Figure 6 shows that the streamwise shear stress τ_{01} is reduced, and that in this example one would expect the over-all drag of the upper surface of the wing to be about three-quarters of the incompressible value. The drag will, however, be increased over the lower surface. The crosswise shear stress τ_{02} on the upper surface is increased to nearly 1·5 times the incompressible value; this figure will be more than 2 over much of the lower surface.

The values just given are for $C = 1$ and $Pr = 1$. For an aeroplane flying in the stratosphere at Mach 2 with zero heat transfer, the value of C is about 0·89; on the other hand, C may have a value a little greater than unity under conditions experienced in a supersonic wind tunnel (Chapman & Rubesin 1949). These values will not affect the general nature of our conclusions.

Equation	Replace		by
(20)	—	Unchanged	—
(21)	—	Unchanged	—
(27)	Index 6	—	$6\bar{r}$
	Index 3	—	$7-4\bar{r}$
(28)	Index 11	—	$9+2\bar{r}$
	q	—	Q
(29)	Index 10	—	$9+\bar{r}$
	q	—	Q
(30)	a_0^2/a_e^2	—	$(a_0/a_e)^{2\bar{r}}$
(31)	Index 8	—	$7+\bar{r}$
(32)	—	Unchanged	—
(33)	a_0/a_e	—	$(a_0/a_e)^{\bar{r}}$

TABLE 1. Effect of non-unit Prandtl number

The effect of Prandtl number may be roughly assessed by consideration of the alterations made in § 6. We take $Pr = 0{\cdot}72$ and so $\bar{r} = 0{\cdot}85$. We have already seen that at moderate Mach numbers the ratio a_e/a_0 is near to unity; indeed, figures 4 and 5 show that the effect of putting this ratio equal to unity is very small. Hence the small changes in the powers to which a_e/a_0 is raised, as listed in table 1, will have little effect, and the main change will come from the change in q.

Since $\bar{r} = 0{\cdot}85$ and a_e/a_0 is near to 1, the last term in equation (35) is small, but the alteration in the second term may be significant. Thus, at $M_0 = 2$, q will change from 1·80 to 1·68, so that the apparent increase in pressure gradient due to compressibility is not quite so large as appeared for $Pr = 1$, and the risk of separation is reduced a little.

It also follows from § 6 that the change in Prandtl number has little effect on τ_{01} and so on the over-all drag, but that the crosswise skin friction τ_{02} and the angle β are not as large as had previously been estimated.

8. Conclusions

It is possible in the case of small cross-flow, unit Prandtl number and zero heat transfer to correlate a compressible three-dimensional laminar boundary layer with an incompressible three-dimensional laminar boundary layer. This makes it possible to make use of the approximate methods which have been developed for incompressible flow with small cross-flow.

The general effect of compressibility at moderate Mach numbers on the directions of the limiting streamlines and so on separation is approximately the same as if the pressure gradients had been multiplied by $1+\frac{1}{2}(\gamma-1)M_0^2$. Alternatively we may say that if in an incompressible flow the pressure gradients must be less than a certain figure in order to avoid separation, this figure must be divided by $1+\frac{1}{2}(\gamma-1)M_0^2$ in order to avoid separation in compressible flow. Since for $M_0 = 2$ this factor is equal to 1·80, it can be seen that the effect may be severe.

Compressibility causes a reduced drag on the upper surface of a wing of the type considered here, but an increased drag on the lower surface. The crosswise skin friction is in general increased on both surfaces of the wing.

The limitation to unit Prandtl number has only a small effect on the drag, but it changes the factor $1+\frac{1}{2}(\gamma-1)M_0^2$ given above to $1+\frac{1}{2}Pr^{\frac{1}{2}}(\gamma-1)M_0^2$, and this minimizes the severity of the apparent increase in pressure gradients due to compressibility. For instance, it reduces the factor 1·80 above to 1·68.

Appendix

Details of the transformation

Equation (5) shows that there exists a stream function ψ such that

$$r\rho u = \rho_0 \frac{\partial\psi}{\partial\zeta}, \quad r\rho w = -\rho_0 \frac{\partial\psi}{\partial s}.$$

We first make the transformation

$$Z = \frac{a_e}{a_0}\int_0^{\zeta}\frac{\rho}{\rho_0}d\zeta, \quad \bar{S} = s,$$

the notation $\bar{S}$ being introduced to avoid confusion. $\psi(s,\zeta)$ transforms into $\psi(\bar{S},Z)$, and we note that $\partial\psi/\partial s$ is *not* the same as $\partial\psi/\partial\bar{S}$.

We have
$$u = \frac{a_e}{a_0 r}\frac{\partial\psi}{\partial Z}, \quad w = -\frac{\rho_0}{\rho r}\left\{\frac{\partial\psi}{\partial\bar{S}} + \frac{\partial\psi}{\partial Z}\frac{\partial Z}{\partial s}\right\}. \tag{A 1}$$

Denoting $\partial\psi/\partial Z$ by ψ_Z, etc., we have

$$\frac{\partial u}{\partial s} = \frac{1}{a_0 r}\psi_Z\frac{\partial a_e}{\partial\bar{S}} + \frac{a_e}{a_0 r}\psi_{Z\bar{S}} + \frac{a_e}{a_0 r}\frac{\partial Z}{\partial s}\psi_{ZZ} - \frac{a_e}{a_0 r^2}\frac{\partial r}{\partial\bar{S}}\psi_Z,$$

$$\frac{\partial u}{\partial\zeta} = \frac{a_e^2\rho}{a_0^2\rho_0 r}\psi_{ZZ}.$$

We also have
$$\mu\frac{\partial u}{\partial\zeta} = C\frac{\mu_0}{r}\frac{T}{T_0}\frac{a_e^2\rho}{a_0^2\rho_0}\psi_{ZZ} = \frac{Cpa_e^2\mu_0}{p_0 a_0^2 r}\psi_{ZZ},$$

using the relation (6) and the equation $p = R\rho T$.

Hence, assuming as usual that p is independent of ζ or Z, we obtain

$$\frac{\partial}{\partial\zeta}\left(\mu\frac{\partial\mu}{\partial\zeta}\right) = \frac{Cp}{p_0}\frac{a_e^3}{a_0^3}\frac{\mu_0\rho}{\rho_0 r}\psi_{ZZZ} = \nu_0\frac{Cp}{p_0}\frac{a_e^3\rho}{a_0^3 r}\psi_{ZZZ}.$$

In the external isentropic flow, we also have Bernoulli's equation

$$a_e^2 + \tfrac{1}{2}(\gamma-1)u_e^2 = \text{const.} = a_0^2 + \tfrac{1}{2}(\gamma-1)u_0^2, \tag{A 2}$$

since $v_e = 0$ in the main stream.

Hence
$$a_e\frac{\partial a_e}{\partial\bar{S}} = -\tfrac{1}{2}(\gamma-1)u_e\frac{\partial u_e}{\partial\bar{S}}. \tag{A 3}$$

Since the pressure in the boundary layer is constant at a given station, we have

$$\frac{\rho_e}{\rho} = \frac{T}{T_e} = 1 + \frac{\gamma-1}{2a_e^2}(u_e^2 - u^2)$$

by equation (7), neglecting v^2.

Substituting in the equation of motion (2), we obtain

$$\frac{1}{r^2}\left\{\psi_Z\psi_{Z\bar{S}} - \frac{1}{r}\frac{\partial r}{\partial \bar{S}}\psi_Z^2 - \psi_{\bar{S}}\psi_{ZZ}\right\} = \frac{a_0^2}{a_e^2}u_e\frac{\partial u_e}{\partial \bar{S}}\left\{1 + \frac{\gamma-1}{2a_e^2}u_e^2\right\} + \frac{\nu_0 a_e}{a_0 r}\frac{Cp}{p_0}\psi_{ZZZ}. \quad (A\,4)$$

If we use equations (10) and (A 3), we find

$$U_e\frac{\partial U_e}{\partial \bar{S}} = \frac{a_0^2}{a_e^2}u_e\frac{\partial u_e}{\partial \bar{S}}\left\{1 + \frac{\gamma-1}{2a_e^2}u_e^2\right\}. \quad (A\,5)$$

Further, if we note that $$\frac{p}{p_0} = \left(\frac{a_e}{a_0}\right)^{2\gamma/(\gamma-1)},$$

and write $$S = C\int\left(\frac{a_e}{a_0}\right)^{(3\gamma-1)/(\gamma-1)} d\bar{S},$$

and also use (A 5), we find that equation (A 4) becomes

$$\frac{1}{r^2}\left\{\psi_Z\psi_{ZS} - \frac{1}{r}\frac{\partial r}{\partial S}\psi_Z^2 - \psi_S\psi_{ZZ}\right\} = U_e\frac{\partial U_e}{\partial S} + \frac{\nu_0}{r}\psi_{ZZZ}.$$

Finally, we write in this equation

$$U = \frac{1}{r}\psi_Z, \quad W = -\frac{1}{r}\psi_S, \quad (A\,6)$$

thus satisfying an equation of the same form as (5) with constant ρ. We obtain

$$U\frac{\partial U}{\partial S} + W\frac{\partial U}{\partial Z} = U_e\frac{\partial U_e}{\partial S} + \nu_0\frac{\partial^2 U}{\partial Z^2}. \quad (A\,7)$$

The above procedure is almost identical with that of Stewartson (1949).

We now substitute in equation (3), writing

$$V = v,$$

$$q = 1 + \tfrac{1}{2}(\gamma-1)M_0^2 = \frac{1}{a_0^2}\{a_0^2 + \tfrac{1}{2}(\gamma-1)u_0^2\} = \frac{1}{a_0^2}\{a_e^2 + \tfrac{1}{2}(\gamma-1)u_e^2\}$$

by equation (A 2). The result is

$$\frac{1}{r}\psi_Z V_S - \frac{1}{r}V_Z\psi_S + \frac{1}{r^2}\psi_Z\frac{\partial r}{\partial S}V = \frac{\kappa}{C}\left(\frac{a_0}{a_e}\right)^{(4\gamma-2)/(\gamma-1)}\left(U_e^2 - \frac{1}{r^2}\psi_Z^2\right)q + \nu_0 V_{ZZ}.$$

Using equations (12) and (A 6), we find

$$U\frac{\partial V}{\partial S} + W\frac{\partial V}{\partial Z} + \frac{UV}{r}\frac{\partial r}{\partial S} = K(U_e^2 - U^2) + \nu_0\frac{\partial^2 V}{\partial Z^2}.$$

Note that, by equations (A 1) and (A 6), we have $u = (a_e/a_0)U$, so that when $u = u_e$ we have $U = U_e$ by equation (10).

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